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THERMAL STRESSES IN A THICK PLATE†

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Abstract—A three-dimensional series solution is obtained for elastic plates subjected to general temperature distribution. The solution satisfies the field equations of linear thermoelasticity and the boundary condition of vanishing stresses on the flat surfaces, but in general not the edge conditions. In the series solution the first term represents the classical thin plate theory and the subsequent correcting terms involve successively higher derivatives of the temperature function. General terms of the series solution are derived for the case of linear temperature distribution $T(x, y, z) = T_0(x, y)+zT_1(x, y)$.

INTRODUCTION

THERMAL stresses in thin plates are generally considered to be well established [1–3], on the other hand investigations pertaining to thick plates appear to be very limited. Sokolnikoff [4] obtained an explicit solution for the stress component τ_{zz}, together with τ_{xz} and τ_{yz} expressed in terms of τ_{zz}; the other three stress components τ_{xx}, τ_{yy}, and τ_{xy} were ignored in the paper. The derivation was rigorous but lengthy, and the final results were not in a convenient form for direct application. Gatewood [5] extended the work of [4] to include calculations of τ_{xx}, τ_{yy}, τ_{xy}, also expressed in terms of τ_{zz}. Some applications and numerical results were given.

In the present paper a series solution, in the same spirit of [4, 5], is obtained by a different method of approach. The results reached for all stress components are expressed explicitly in terms of the prescribed temperature load. The temperature distribution is assumed to be expressible in the form

$$T(x, y, z) = \sum_{k=0}^{N} z^k T_k(x, y). \tag{1}$$

The solution satisfies the field equations of linear thermoelasticity and the boundary condition of vanishing stresses on the flat surfaces. Accommodations are provided for handling the edge conditions in each individual problem.

In the series solution the first term represents the classical thin plate theory and the subsequent correcting terms involve successively higher derivatives of the temperature function. General terms for the stresses are derived for the case of linear temperature distribution

$$T(x, y, z) = T_0(x, y)+zT_1(x, y). \tag{2}$$

The displacements are determined as usual from the stresses.

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The same type of approach was first used by Boley [6] for the thermoelastic solution of beams. Furthermore, it can be shown that the present plate solution may be reduced, under appropriate conditions, to the corresponding beam solution [6].

DERIVATION

The governing field equations to be satisfied by the series solution are [1, 3]

$$\begin{aligned} \tau_{ij,j} &= 0 \\ \nabla_1^2\tau_{ij}+\frac{1}{1+v}S_{,ij} &= -\frac{\alpha E}{1+v}\left(\frac{1+v}{1-v}\delta_{ij}\nabla_1^2 T+T_{,ij}\right) \end{aligned} \tag{3}$$

where

$$\nabla_1^2 = \nabla^2+\frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \tag{4}$$

$$S = \tau_{xx}+\tau_{yy}+\tau_{zz}. \tag{5}$$

The usual rules for index notation apply. The boundary conditions to be satisfied by the solution are

$$z = \pm c\colon \qquad \tau_{zz} = \tau_{zx} = \tau_{zy} = 0. \tag{6}$$

The stress components given by the classical thin plate theory are [1–3]

$$\begin{aligned} \tau_{xx} &= E\alpha\left\{-\phi_{,yy}+\frac{1}{1-v}z(\psi_{,xx}+v\psi_{,yy}-M_T)+\frac{1}{1-v}(N_T+zM_T-T)\right\} \\ \tau_{zz} &= 0 \\ \tau_{xy} &= E\alpha\{\phi_{,xy}+z\psi_{,xy}\} \\ \tau_{xz} &= \frac{E\alpha}{1-v}\{-\tfrac{1}{2}(z^2-c^2)(\nabla^2\psi_{,x}-M_{T,x})\} \end{aligned} \tag{7}$$

with τ_{yy} and τ_{yz} to be obtained from τ_{xx} and τ_{xz}, respectively, by interchanging x and y. In equations (7) the functions $\phi(x, y)$ and $\psi(x, y)$ are determined from

$$\nabla^4\phi = \nabla^2 N_T, \qquad \nabla^4\psi = \nabla^2 M_T \tag{8}$$

and with N_T and M_T defined by

$$\begin{aligned} N_T(x, y) &= \frac{1}{2c}\int_{-c}^{c} T\,\mathrm{d}z \\ M_T(x, y) &= \frac{3}{2c^3}\int_{-c}^{c} Tz\,\mathrm{d}z. \end{aligned} \tag{9}$$

A comma in equations (7) indicates partial differentiation.

It can be readily checked that the stress components given in equations (7) do not satisfy completely all equations (3) and (6). These expressions (7) will be taken as first terms of the series solution, to be designated as $\{\ \}_0$ terms. The next terms $\{\ \}_1$ are derived by assuming appropriate expressions for each stress components with unknown coefficients, which are determined by using equations (3) and (6); in the calculation derivatives of N_T and M_T higher than the third order are neglected. The rest of the terms are derived by repeating the procedure, and in deriving terms $\{\ \}_n$ the derivatives of N_T and M_T higher than the $(2n+1)$th order are neglected. It may be mentioned that the same method of approach have been used in the earlier papers [7, 8] for isothermal case.

Once the stress components are known, the displacement components may be determined through the relations

$$\begin{aligned} e_{ij} &= \frac{1}{E}[(1+\nu)\tau_{ij}-\nu\delta_{ij}S]+\delta_{ij}\alpha T \\ e_{ij} &= \tfrac{1}{2}(u_{i,j}+u_{j,i}). \end{aligned} \tag{10}$$

RESULTS

(a) *Temperature distribution*

The temperature distribution $T(x, y, z)$ is assumed to have the form of equation (1). Substitution of (1) into (9) results

$$\begin{aligned} N_T &= T_0+\frac{c^2}{3}T_2+\ \dots\ +\frac{c^i}{i+1}T_i+\ \dots (i = \text{even}) \\ M_T &= T_1+\frac{3}{5}c^2T_3+\ \dots\ +\frac{3}{j+2}c^{j-1}T_j+\ \dots (j = \text{odd}). \end{aligned} \tag{11}$$

The function $T(x, y, z)$ may be rearranged as

$$T = N_T+zM_T+T_H(x, y, z) \tag{12}$$

where

$$T_H = \sum_{i=2}^{N}\left(z^i-\frac{c^i}{i+1}\right)T_i+\sum_{j=3}^{N}\left(z^j-\frac{3}{j+2}c^{j-1}z\right)T_j \tag{13}$$

with i = even and j = odd integers only. It is obvious that in the case of linear temperature distribution (2), we have

$$T_H = 0, \qquad N_T = T_0, \qquad M_T = T_1 \tag{14}$$

and equation (12) is identical to equation (2).

(b) *Formulas for the part $T = N_T + zM_T$*

The series solution for stress components are obtained as follows.

$$
\begin{aligned}
\frac{1-\nu}{E\alpha}\tau_{xx} &= \{-(1-\nu)\phi_{,yy} + z(\psi_{,xx} + \nu\psi_{,yy} - M_T)\}_0 + \left\{\nabla^2\phi_{,yy}\left(\frac{\nu}{2}\frac{1-\nu}{1+\nu}z^2\right)\right. \\
&\quad + N_{T,xx}\left(-\frac{z^2}{2}+\frac{c^2}{6}\right) + \nu N_{T,yy}\left(-\frac{z^2}{1+\nu}+\frac{c^2}{6}\right) - \nabla^2\psi_{,xx}\left(\frac{2-\nu}{6}z^3\right) \\
&\quad \left. + M_{T,xx}\left(\frac{1-\nu}{6}z^3 + \frac{c^2z}{10}\right) + \nu M_{T,yy}\left(-\frac{z^3}{6}+\frac{1}{10}c^2z\right)\right\}_1 \\
&\quad + \left\{\nabla^2 N_{T,xx}\left(\frac{z^4}{12} - \frac{c^2z^2}{6} + \frac{7}{180}c^4\right) + \nu\nabla^2 N_{T,yy}\left(\frac{z^4}{24} - \frac{c^2z^2}{12} - \frac{c^4}{360}\right)\right. \\
&\quad \left. + \nabla^2 M_{T,xx}\left(\frac{z^5}{60} - \frac{c^2z^3}{30} - \frac{9}{700}c^4z\right) + \nu\nabla^2 M_{T,yy}\left(\frac{z^5}{120} - \frac{c^2z^3}{60} + \frac{19}{4200}c^4z\right)\right\}_2 + \ldots \\
\frac{1-\nu}{E\alpha}\tau_{zz} &= \{0\}_0 + \{0\}_1 + \left\{\nabla^4 N_T\left(-\frac{z^4}{24} + \frac{c^2z^2}{12} - \frac{c^4}{24}\right) + \nabla^4 M_T\left(-\frac{z^5}{120} + \frac{c^2z^3}{60} - \frac{c^4z}{120}\right)\right\}_2 + \ldots \\
\frac{1-\nu}{E\alpha}\tau_{xy} &= (1-\nu)\{\phi_{,xy} + z\psi_{,xy}\}_0 + \left\{\nabla^2\phi_{,xy}\left(-\frac{\nu}{2}\frac{1-\nu}{1+\nu}z^2\right) + N_{T,xy}\left(-\frac{1}{2}\frac{1-\nu}{1+\nu}z^2 + \frac{1-\nu}{6}c^2\right)\right. \\
&\quad \left. + \nabla^2\psi_{,xy}\left(-\frac{2-\nu}{6}z^3\right) + M_{T,xy}\left(\frac{z^3}{6} + \frac{1-\nu}{10}c^2z\right)\right\}_1 \\
&\quad + \left\{\nabla^2 N_{T,xy}\left(\frac{2-\nu}{24}z^4 - \frac{2-\nu}{12}c^2z^2 + \frac{14+\nu}{360}c^4\right)\right. \\
&\quad \left. + \nabla^2 M_{T,xy}\left(\frac{2-\nu}{120}z^5 - \frac{2-\nu}{60}c^2z^3 + \frac{54-19\nu}{4200}c^4z\right)\right\}_2 + \ldots \\
\frac{1-\nu}{E\alpha}\tau_{xz} &= \left\{(\nabla^2\psi_{,x} - M_{T,x})\left(-\frac{z^2}{2}+\frac{c^2}{2}\right)\right\}_0 + \left\{\nabla^2 N_{T,x}\left(\frac{z^3}{6} - \frac{c^2z}{6}\right)\right. \\
&\quad \left. + \nabla^2 M_{T,x}\left(\frac{z^4}{24} - \frac{c^2z^2}{20} + \frac{c^4}{120}\right)\right\}_1 + \left\{\nabla^4 N_{T,x}\left(-\frac{z^5}{60} + \frac{c^2z^3}{18} - \frac{7}{180}c^4z\right)\right. \\
&\quad \left. + \nabla^4 M_{T,x}\left(-\frac{z^6}{360} + \frac{c^2z^4}{120} - \frac{9}{1400}c^4z^2 + \frac{11}{12{,}600}c^6\right)\right\}_2 + \ldots
\end{aligned}
\tag{15}
$$

In the above equations ϕ and ψ are determined from equations (8) subjected to prescribed edge conditions.

The general terms $\{\ \}_n$ for the stress components have the form

$$\frac{1-\nu}{E\alpha}\tau_{xx}: \sum_{m=0}^{n} \{\nabla^{2(n-1)}(a_{mn}N_{T,xx}+\nu b_{mn}N_{T,yy})c^{2m}z^{2(n-m)}$$

$$+\nabla^{2(n-1)}(d_{mn}M_{T,xx}+\nu e_{mn}M_{T,yy})c^{2m}z^{2n-2m+1}\}_n$$

$$\frac{1-\nu}{E\alpha}\tau_{zz}: \sum_{m=0}^{n} \{\nabla^{2n}(f_{mn}N_T)c^{2m}z^{2(n-m)}+\nabla^{2n}(g_{mn}M_T)c^{2m}z^{2n-2m+1}\}_n$$

$$\frac{1-\nu}{E\alpha}\tau_{xy}: \sum_{m=0}^{n} \{\nabla^{2(n-1)}(h_{mn}N_{T,xy})c^{2m}z^{2(n-m)}+\nabla^{2(n-1)}(i_{mn}M_{T,xy})c^{2m}z^{2n-2m+1}\}_n$$

$$\frac{1-\nu}{E\alpha}\tau_{xz}: \left\{\sum_{m=0}^{n+1} \nabla^{2n}(j_{mn}N_{T,x})c^{2m}z^{2n-2m+1}+\sum_{m=0}^{n+1} \nabla^{2n}(k_{mn}M_{T,x})c^{2m}z^{2(n-m+1)}\right\}_n. \tag{16}$$

The following formulas give the numerical coefficients for the $(n+1)$th term as functions of the corresponding coefficients of the nth term. The first group of formulas hold for $m = 0, 1, \ldots, n$:

$$\begin{aligned}
a_{m(n+1)} &= -\frac{1}{2(n-m+1)(2n-2m+1)}\left[a_{mn}+\frac{a_{mn}+\nu b_{mn}+f_{mn}}{1+\nu}\right]\\
b_{m(n+1)} &= -\frac{b_{mn}}{2(n-m+1)(2n-2m+1)}\\
f_{m(n+1)} &= -\frac{j_{mn}}{2(n-m+1)}\\
h_{m(n+1)} &= -\frac{1}{2(n-m+1)(2n-2m+1)}\left[h_{mn}+\frac{a_{mn}+\nu b_{mn}+f_{mn}}{1+\nu}\right]\\
j_{m(n+1)} &= -\frac{a_{m(n+1)}}{2n-2m+3}\\
d_{m(n+1)} &= -\frac{1}{2(n-m+1)(2n-2m+3)}\left[d_{mn}+\frac{d_{mn}+\nu e_{mn}+g_{mn}}{1+\nu}\right]\\
e_{m(n+1)} &= -\frac{e_{mn}}{2(n-m+1)(2n-2m+3)}\\
g_{m(n+1)} &= -\frac{k_{mn}}{2n-2m+3}\\
i_{m(n+1)} &= -\frac{1}{2(n-m+1)(2n-2m+3)}\left[i_{mn}+\frac{d_{mn}+\nu e_{mn}+g_{mn}}{1+\nu}\right]\\
k_{m(n+1)} &= -\frac{d_{m(n+1)}}{2(n-m+2)}.
\end{aligned} \tag{17}$$

For $m = n+1$ and $m = n+2$, the following formulas must be used:

$$
\begin{aligned}
f_{(n+1)(n+1)} &= -\sum_{m=0}^{n} f_{m(n+1)} \\
j_{(n+1)(n+1)} &= -\sum_{m=0}^{n} j_{m(n+1)} \\
a_{(n+1)(n+1)} &= -j_{(n+1)(n+1)} \\
b_{(n+1)(n+1)} &= f_{(n+1)(n+1)} - j_{(n+1)(n+1)} \\
h_{(n+1)(n+1)} &= -j_{(n+1)(n+1)} - \nu b_{(n+1)(n+1)} \\
g_{(n+1)(n+1)} &= -\sum_{m=0}^{n} g_{m(n+1)} \\
k_{(n+1)(n+1)} &= -\sum_{m=0}^{n} \frac{3(n-m+2)}{2n-2m+5} k_{m(n+1)} \\
k_{(n+2)(n+1)} &= -\sum_{m=0}^{n+1} k_{m(n+1)} \\
d_{(n+1)(n+1)} &= -2k_{(n+1)(n+1)} \\
e_{(n+1)(n+1)} &= g_{(n+1)(u+1)} - 2k_{(n+1)(n+1)} \\
i_{(n+1)(n+1)} &= -\nu e_{(n+1)(n+1)} - 2k_{(n+1)(n+1)}.
\end{aligned}
\tag{18}
$$

The series solution for displacement components are found as follows:

$$
\begin{aligned}
\frac{1-\nu}{\alpha}u_x &= (1-\nu)\{(1+\nu)(\phi_{,x} + z\psi_{,x}) + P(x, y)\}_0 \\
&\quad + (1+\nu)\left\{\frac{1-\nu}{2}\left[\nabla^2\phi_{,x}\left(-\frac{\nu}{1+\nu}z^2\right) + N_{T,x}\left(-\frac{z^2}{1+\nu} + \frac{c^2}{3}\right)\right] + \nabla^2\psi_{,x}\left(-\frac{2-\nu}{6}z^3\right)\right. \\
&\quad \left. + M_{T,x}\left(\frac{z^3}{6} + \frac{1-\nu}{10}c^2 z\right)\right\}_1 + (1+\nu)\left\{\nabla^2 N_{T,x}\left(\frac{2-\nu}{24}z^4 - \frac{2-\nu}{12}c^2z^2 + \frac{14+\nu}{360}c^4\right)\right. \\
&\quad \left. + \nabla^2 M_{T,x}\left(\frac{2-\nu}{120}z^5 - \frac{2-\nu}{60}c^2z^3 + \frac{54-19\nu}{4200}c^4 z\right)\right\}_2 + \dots \\
\frac{1-\nu}{\alpha}u_z &= \{-(1-\nu^2)\psi\}_0 + \left\{(1-\nu)z(\nu\nabla^2\phi + N_T) + (1+\nu)\left[\nabla^2\psi\left(-\frac{\nu}{2}z^2 + c^2\right)\right.\right. \\
&\quad \left.\left. + M_T\left(\frac{z^2}{2} - \frac{11-\nu}{10}c^2\right)\right]\right\}_1 + (1+\nu)\left\{\nu\nabla^2 N_T\left(\frac{z^3}{6} - \frac{c^2 z}{6}\right)\right. \\
&\quad \left. + \nabla^2 M_T\left(\frac{\nu}{24}z^4 - \frac{\nu}{20}c^2z^2 + \frac{16+19\nu}{4200}c^4\right)\right\}_2 + \dots
\end{aligned}
\tag{19}
$$

The formula for u_y may be obtained from u_x by interchanging x and y, except that the function $P(x, y)$ in terms $\{\ \}_0$ is changed to $Q(x, y)$. P and Q are the real and imaginary parts of an analytic function of complex variable,

$$f(z) = P + iQ \tag{20}$$

with $z = x + iy$, and

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} = N_T - \nabla^2 \phi. \tag{21}$$

(c) *Formulas for the term* $T = [z^i - (c^i/i+1)]T_i$

The series solution of stress components for this typical temperature term, with $i = 2, 4, 6, \ldots$, are found to be

$$\begin{aligned}
\frac{(i+1)(1-\nu)}{E\alpha}\tau_{xx} &= T_i\{-(i+1)z^i + c^i\}_0 + \left\{T_{i,xx}\left[\frac{z^{i+2}}{i+2} - \frac{c^i z^2}{2} + \frac{i(i+5)}{6(i+2)(i+3)}c^{i+2}\right]\right. \\
&\quad \left. + \nu T_{i,yy}\left[-\frac{i}{3(i+3)}c^{i+2}\right]\right\}_1 + \left\{\nabla^2 T_{i,xx}\left[-\frac{z^{i+4}}{(i+2)(i+3)(i+4)} + \frac{c^i z^4}{24}\right.\right. \\
&\quad \left. + \frac{i(i-1)}{12(i+2)(i+3)}c^{i+2}z^2 - \frac{i(13i^2+96i+131)}{360(i+3)(i+4)(i+5)}c^{i+4}\right] \\
&\quad \left. + \frac{\nu}{6}\nabla^2 T_{i,yy}\left[\frac{i}{i+3}c^{i+2}z^2 - \frac{i(7i^2+69i+164)}{15(i+3)(i+4)(i+5)}c^{i+4}\right]\right\}_2 + \ldots \\
\frac{(i+1)(1-\nu)}{E\alpha}\tau_{zz} &= \{0\}_0 + \nabla^2 T_i\left\{-\frac{z^{i+2}}{i+2} + \frac{c^i z^2}{2} - \frac{i}{2(i+2)}c^{i+2}\right\}_1 \\
&\quad + \nabla^4 T_i\left\{\frac{z^{i+4}}{(i+2)(i+3)(i+4)} - \frac{c^i z^4}{24} + \frac{i(i+5)}{12(i+2)(i+3)}c^{i+2}z^2\right. \\
&\quad \left. - \frac{i(i+7)}{24(i+3)(i+4)}c^{i+4}\right\}_2 + \ldots \\
\frac{(i+1)(1-\nu)}{E\alpha}\tau_{xy} &= \{0\}_0 + T_{i,xy}\left\{\frac{z^{i+2}}{i+2} - \frac{c^i z^2}{2} + \frac{i}{6}\left[\frac{i+5}{(i+2)(i+3)} + \frac{2\nu}{i+3}\right]c^{i+2}\right\}_1 \\
&\quad + \nabla^2 T_{i,xy}\left\{-\frac{z^{i+4}}{(i+2)(i+3)(i+4)} + \frac{c^i z^4}{24} + \frac{i}{12}\left[\frac{i-1}{(i+2)(i+3)} - \frac{2\nu}{i+3}\right]c^{i+2}z^2\right. \\
&\quad \left. - \frac{i}{360}\left[\frac{13i^2+96i+131}{(i+3)(i+4)(i+5)} - \frac{4(7i^2+69i+164)\nu}{(i+3)(i+4)(i+5)}\right]c^{i+4}\right\}_2 + \ldots \\
\frac{(i+1)(1-\nu)}{E\alpha}\tau_{xz} &= T_{i,x}\{z^{i+1} - c^i z\}_0 + \nabla^2 T_{i,x}\left\{-\frac{z^{i+3}}{(i+2)(i+3)} + \frac{c^i z^3}{6} - \frac{i(i+5)}{6(i+2)(i+3)}c^{i+2}z\right\}_1 \\
&\quad + \nabla^4 T_{i,x}\left\{\frac{z^{i+5}}{(i+2)(i+3)(i+4)(i+5)} - \frac{c^i z^5}{120} - \frac{i(i-1)}{36(i+2)(i+3)}c^{i+2}z^3\right. \\
&\quad \left. + \frac{i(13i^2+96i+131)}{360(i+3)(i+4)(i+5)}c^{i+4}z\right\}_2 + \ldots
\end{aligned} \tag{22}$$

(d) *Formulas for the term* $T = \left(z^j - \frac{3}{j+2}c^{j-1}z\right)T_j$

The series solution of stress components for this typical temperature term, with $j = 3, 5, 7, \ldots$, are found to be

$$\begin{aligned}
\frac{(j+2)(1-\nu)}{E\alpha}\tau_{xx} = {} & T_j\{-(j+2)z^j + 3c^{j-1}z\}_0 + \left\{T_{j,xx}\left[\frac{z^{j+2}}{j+1} - \frac{c^{j-1}z^3}{2}\right.\right. \\
& \left.\left. + \frac{3(j-1)(j+6)}{10(j+1)(j+4)}c^{j+1}z\right] - \nu T_{j,yy}\left[\frac{j-1}{5(j+4)}c^{j+1}z\right]\right\}_1 \\
& + \left\{\nabla^2 T_{j,xx}\left[-\frac{z^{j+4}}{(j+1)(j+3)(j+4)} + \frac{1}{40}c^{j-1}z^5 - \frac{(j-1)(j+16)}{60(j+1)(j+4)}c^{j+1}z^3\right.\right. \\
& \left. - \frac{(j-1)(j^2-126j-912)}{1400(j+3)(j+4)(j+6)}c^{j+3}z\right] + \frac{\nu}{10}\nabla^2 T_{j,yy}\left[\frac{j-1}{3(j+4)}c^{j+1}z^3\right. \\
& \left.\left. - \frac{(j-1)(9j+64)}{35(j+4)(j+6)}c^{j+3}z\right]\right\}_2 + \ldots \\
\frac{(j+2)(1-\nu)}{E\alpha}\tau_{zz} = {} & \{0\}_0 + \nabla^2 T_j\left\{-\frac{z^{j+2}}{j+1} + \frac{1}{2}c^{j-1}z^3 - \frac{j-1}{2(j+1)}c^{j+1}z\right\}_1 + \nabla^4 T_j \\
& \times\left\{\frac{z^{j+4}}{(j+1)(j+3)(j+4)} - \frac{c^{j-1}z^5}{40} + \frac{(j-1)(j+6)}{20(j+1)(j+4)}c^{j+1}z^3\right. \\
& \left. - \frac{(j-1)(j+8)}{40(j+3)(j+4)}c^{j+3}z\right\}_2 + \ldots \\
\frac{(j+2)(1-\nu)}{E\alpha}\tau_{xy} = {} & \{0\}_0 + T_{j,xy}\left\{\frac{z^{j+2}}{j+1} - \frac{1}{2}c^{j-1}z^3 + \frac{j-1}{10}\left[\frac{3(j+6)}{(j+1)(j+4)} + \frac{2\nu}{j+4}\right]c^{j+1}z\right\}_1 \\
& + \nabla^2 T_{j,xy}\left\{-\frac{z^{j+4}}{(j+1)(j+3)(j+4)} + \frac{1}{40}c^{j-1}z^5 - \frac{j-1}{60}\left[\frac{j+16}{(j+1)(j+4)}\right.\right. \\
& \left.\left. + \frac{2\nu}{j+4}\right]c^{j+1}z^3 - \frac{j-1}{1400}\left[\frac{j^2-126j-912}{(j+3)(j+4)(j+6)} - \frac{4(9j+64)\nu}{(j+3)(j+6)}\right]c^{j+3}z\right\}_2 + \ldots \\
\frac{(j+2)(1-\nu)}{E\alpha}\tau_{xz} = {} & T_{j,x}\left\{\frac{j+2}{j+1}z^{j+1} - \frac{3}{2}c^{j-1}z^2 + \frac{j-1}{2(j+1)}c^{j+1}\right\}_0 + \nabla^2 T_{j,x}\left\{-\frac{z^{j+3}}{(j+1)(j+3)}\right. \\
& \left. + \frac{1}{8}c^{j-1}z^4 - \frac{3(j-1)(j+6)}{20(j+1)(j+4)}c^{j+1}z^2 + \frac{(j-1)(j+8)}{40(j+3)(j+4)}c^{j+3}\right\}_1 + \nabla^4 T_{j,x} \\
& \times\left\{\frac{z^{j+5}}{(j+1)(j+3)(j+4)(j+5)} - \frac{1}{240}c^{j+3}z^6 + \frac{(j-1)(j+16)}{240(j+1)(j+4)}c^{j+1}z^4\right. \\
& \left. + \frac{(j-1)(j^2-126j-912)}{2800(j+3)(j+4)(j+6)}c^{j+3}z^2 - \frac{(j-1)(3j^2-22j-360)}{8400(j+4)(j+5)(j+6)}c^{j+5}\right\}_2 + \ldots
\end{aligned} \tag{23}$$

The superposition of the above results according to equations (12) and (13) gives the complete solution for a thick plate. In the case of linear temperature distribution (2), the solution given in part (b) alone represents the complete solution.

DISCUSSION

(1). If the temperature distribution is linear, as given in equation (2), application of equations (16), (17), (18) together with equations (15) gives

$$\frac{1-\nu}{E\alpha}\tau_{zz} = \frac{(z^2-c^2)^2}{24}\left\{-\nabla^4 T_0 - \frac{z}{5}\nabla^4 T_1 + \frac{z^2-3c^2}{15}\nabla^6 T_0 + \frac{5z^3-11c^2z}{525}\nabla^6 T_1 \right.$$
$$\left. -\frac{9z^4-66c^2z^2+41c^4}{5040}\nabla^8 T_0 - \frac{25z^5-130c^2z^3+153c^4z}{126{,}000}\nabla^8 T_1 + \dots \right\}. \quad (24)$$

This equation checks with Sokolnikoff's result except an opposite sign for the leading term $\nabla^4 T_0$. It reduces to Boley's beam solution [1] when appropriate simplification is made.

(2). Under appropriate conditions, for example take the linear temperature distribution, if we substitute

$$\nu = 0, \qquad \frac{d^2\phi}{dx^2} = T_0, \qquad \frac{d^2\psi}{dx^2} = T_1,$$

and consider that all quantities are independent of y, then equations (15) reduce to Boley's beam solution, as appeared in Tables 10.2 and 10.3 of Ref. [1]. Similarly for the case of

$$T = z^2 T_2(x) + z^3 T_3(x)$$

then the superposition of equations (15) and (27) reduces to Boley's results, Table 10.1 of Ref. [1]. Notice that equations (27) appear in the next item of discussion.

(3). To illustrate the use of the formulas given in parts (c) and (d), let the temperature distribution

$$T(x, y, z) = T_0 + zT_1 + z^2 T_2 + z^3 T_3. \quad (25)$$

Then the terms in equation (12) have the values

$$N_T = T_0 + \frac{c^2}{3}T_2$$
$$M_T = T_1 + \tfrac{3}{5}c^2 T_3$$
$$T_H = \left(z^2 - \frac{c^2}{3}\right)T_2 + (z^3 - \tfrac{3}{5}c^2 z)T_3 \quad (26)$$

and the stress components due to the part $T = T_H$ become

$$\frac{1-\nu}{E\alpha}\tau_{xx} = \left\{T_2\left(-z^2+\frac{c^2}{3}\right) + T_3\left(-z^3+\frac{3}{5}c^2 z\right)\right\}_0 + \left\{T_{2,xx}\left(\frac{z^4}{12}-\frac{c^2z^2}{6}+\frac{7}{180}c^4\right)\right.$$
$$\left. +\nu T_{2,yy}\left(-\frac{2c^4}{45}\right) + T_{3,xx}\left(\frac{z^5}{20}-\frac{c^2z^3}{10}+\frac{27}{700}c^4 z\right) + \nu T_{3,yy}\left(-\frac{2}{175}c^4 z\right)\right\}_1$$
$$+\left\{\nabla^2 T_{2,xx}\left(-\frac{z^6}{360}+\frac{c^2z^4}{72}+\frac{1}{360}c^4z^2-\frac{5}{1512}c^6\right)\right.$$
$$+\nu\nabla^2 T_{2,yy}\left(\frac{c^4z^2}{45}-\frac{11}{945}c^6\right) + \nabla^2 T_{3,xx}\left(-\frac{z^7}{840}+\frac{1}{200}c^2z^5-\frac{19}{4200}c^4z^3\right.$$
$$\left.\left. +\frac{61}{63{,}000}c^6 z\right) + \nu\nabla^2 T_{3,yy}\left(\frac{1}{525}c^4z^3-\frac{13}{7875}c^6 z\right)\right\}_2 + \dots$$

$$\begin{aligned}
\frac{1-\nu}{E\alpha}\tau_{zz} = {} & \{0\}_0+\left\{\nabla^2 T_2\left(-\frac{z^4}{12}+\frac{c^2z^2}{6}-\frac{c^4}{12}\right)+\nabla^2 T_3\left(-\frac{z^5}{20}+\frac{c^2z^3}{10}-\frac{c^4z}{20}\right)\right\}_1 \\
& +\left\{\nabla^4 T_2\left(\frac{z^6}{360}-\frac{c^2z^4}{72}+\frac{7}{360}c^4z^2-\frac{c^6}{120}\right)+\nabla^4 T_3\left(\frac{z^7}{840}-\frac{c^2z^5}{200}+\frac{9}{1400}c^4z^3\right.\right. \\
& \left.\left.-\frac{11}{4200}c^6z\right)\right\}_2+\dots \\
\frac{1-\nu}{E\alpha}\tau_{xy} = {} & \{0\}_0+\left\{T_{2,xy}\left(\frac{z^4}{12}-\frac{1}{6}c^2z^2+\frac{7+8\nu}{180}c^4\right)+T_{3,xy}\left(\frac{z^5}{20}-\frac{1}{10}c^2z^3\right.\right. \\
& \left.\left.+\frac{27+8\nu}{700}c^4z\right)\right\}_1+\left\{\nabla^2 T_{2,xy}\left(-\frac{z^6}{360}+\frac{1}{72}c^2z^4+\frac{1-8\nu}{360}c^4z^2-\frac{25-88\nu}{7560}c^6\right)\right. \\
& \left.+\nabla^2 T_{3,xy}\left(-\frac{z^7}{840}+\frac{1}{200}c^2z^5-\frac{19+8\nu}{4200}c^4z^3+\frac{61+104\nu}{63{,}000}c^6z\right)\right\}_2+\dots \\
\frac{1-\nu}{E\alpha}\tau_{xz} = {} & \left\{T_{2,x}\left(\frac{z^3}{3}-\frac{c^2z}{3}\right)+T_{3,x}\left(\frac{z^4}{4}-\frac{3}{10}c^2z^2+\frac{c^4}{20}\right)\right\}_0+\left\{\nabla^2 T_{2,x}\left(-\frac{z^5}{60}\right.\right. \\
& \left.\left.+\frac{c^2z^3}{18}-\frac{7}{180}c^4z\right)+\nabla^2 T_{3,x}\left(-\frac{z^6}{120}+\frac{c^2z^4}{40}-\frac{27}{1400}c^4z^2+\frac{11}{4200}c^6\right)\right\}_1 \\
& +\left\{\nabla^4 T_{2,x}\left(\frac{z^7}{2520}-\frac{c^2z^5}{360}-\frac{1}{1080}c^4z^3+\frac{5}{1512}c^6z\right)+\nabla^4 T_{3,x}\left(\frac{z^8}{6720}-\frac{c^2z^6}{1200}\right.\right. \\
& \left.\left.+\frac{19}{16{,}800}c^4z^4-\frac{61}{126{,}000}c^6z^2+\frac{19}{504{,}000}c^8\right)\right\}_2+\dots
\end{aligned} \tag{27}$$

In the above equations again $\{\,\}_0$ represents the classical thin plate theory and the remaining terms are corrections.

The displacement components corresponding to the stresses of equations (27) are obtained as follows.

$$\begin{aligned}
\frac{1-\nu}{1+\nu}\frac{u_x}{\alpha} = {} & \{0\}_0+\left\{T_{2,x}\left[\frac{z^4}{12}-\frac{c^2z^2}{6}+\frac{7+8}{180}c^4\right]+T_{3,x}\left[\frac{z^5}{20}-\frac{c^2z^3}{10}\right.\right. \\
& \left.\left.+\frac{27+8\nu}{700}c^4z\right]\right\}_1+\left\{\nabla^2 T_{2,x}\left[-\frac{z^6}{360}+\frac{c^2z^4}{72}+\frac{1-8\nu}{360}c^4z^2-\frac{25-88\nu}{7560}c^6\right]\right. \\
& \left.+\nabla^2 T_{3,x}\left[-\frac{z^7}{840}+\frac{c^2z^5}{200}-\frac{19+8\nu}{4200}c^4z^3+\frac{61+104\nu}{63{,}000}c^6z\right]\right\}_2+\dots \\
\frac{1-\nu}{1+\nu}\frac{u_z}{\alpha} = {} & \left\{T_2\left[\frac{z^3}{3}-\frac{c^2z}{3}\right]+T_3\left[\frac{z^4}{4}-\frac{3}{10}c^2z^2+\frac{43-8\nu}{700}c^4\right]\right\}_0 \\
& +\left\{\nabla^2 T_2\left[-\frac{z^5}{60}+\frac{c^2z^3}{18}-\frac{15-8\nu}{180}c^4z\right]+\nabla^2 T_3\left[-\frac{z^6}{120}+\frac{c^2z^4}{40}-\frac{35-8\nu}{1400}c^4z^2\right.\right. \\
& \left.\left.+\frac{269-104\nu}{63{,}000}c^6\right]\right\}_1+\left\{\nabla^4 T_2\left[\frac{z^7}{2520}-\frac{c^2z^5}{360}+\frac{7-8\nu}{1080}c^4z^3-\frac{63-88\nu}{7560}c^6z\right]\right. \\
& \left.+\nabla^4 T_3\left[\frac{z^8}{6720}-\frac{c^2z^6}{1290}+\frac{27-8\nu}{16{,}800}c^4z^4-\frac{165-104\nu}{126{,}000}c^6z^2+\beta c^8\right]\right\}_2+\dots
\end{aligned} \tag{28}$$

In the above equations the rigid-body displacements are ignored and β is a constant which is related to stresses of the term $\{\ \}_3$.

(4). When the present theory is used to solve a particular problem the edge conditions may be handled in the same manner as in classical thin plate theory. Integration of equations (8) will give the functions $\phi(x, y)$ and $\psi(x, y)$ involving arbitrary constants. These functions are then substituted into equations (15) or (19) so as to satisfy the prescribed edge conditions. Naturally the solution of parts (c) and (d) should be superimposed to part (b) in the calculation of edge boundary conditions. An observation of the order of differential equations (8) indicates that four boundary conditions may be satisfied at each edge of the plate.

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Абстракт—Дается трехмерное решение, представленное в рядах, для упругих пластинок, подверженных действию общего поля температуры. Решение удовлетворяет уравнениям поля линейной термоупругости и граничному условию затухания напряжений на плоских поверхностях, но при этом не удавлетворяет краевым условиям. В решению представленному в рядах, первый член оказывается решением классической теории тонких пластинок, а корректирующие члены представляют собой высшие производные функции температуры. Выводятся общие члены решения, представленного в рядах для случая линейного распределения температуры

$$T(x,y,z) = T_0(x,y) + zT_1(x,y).$$